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Anticommuting variables, fermionic path integrals and supersymmetry

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Fermionic Brownian paths are defined as paths in a space parametrised by anticommuting variables. Stochastic calculus for these paths, in conjunction with classical Brownian paths, is described; Brownian paths on supermanifolds are developed and applied to establish a Feynman–Kac formula for the twisted Laplace–Beltrami operator on differential forms taking values in a vector bundle. This formula is used to give a proof of the Atiyah–Singer index theorem which is rigorous while being closely modelled on the supersymmetric proofs in the physics literature.

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1. Introduction

These lectures concern a generalisation of Brownian paths to include fermionic paths, which are paths in spaces parametrised by anticommuting variables. The aim of this work is to provide a rigorous version of some heuristic constructions used with great effect in quantum physics, and in application to geometry.

While fermionic paths do not directly model any physical quantity, they provide a technique for investigating differential operators on spaces of functions of anticommuting variables; since one can obtain an analogue of the Schrödinger representation for fermion operators in quantum physics using just this kind of operator, such paths are useful in path integral quantisation of theories with fermions, as was first observed in a highly original paper of Martin [14]. Additionally, functions on carefully constructed supermanifolds are equivalent to differential forms or spinor fields, which can then be analysed by the geometric fermionic path integration techniques described in these lectures. It is of course possible to handle fermionic quantization without using anticommuting variables; however, in the author's opinion, they are a valuable aid to intuition, particularly in supersymmetric models, and these lectures are presented in the hope of showing that such variables also have analytic power.

Section 2 of these lectures introduces fermionic Brownian paths, and the corresponding Wiener measure. A brief review of conventional stochastic calculus and its use in deriving Feynman–Kac (or path integral) formulae for diffusion operators is then given in section 3. In section 4 it is shown how these methods may be extended to include fermionic paths. Section 5 reviews the standard construction of Brownian paths on manifolds, and extends the construction to paths on carefully chosen supermanifolds, leading to a Feynman–Kac formula for the Laplace–Beltrami operator on twisted differential forms. In the final section these techniques are applied to give a rigorous version of the supersymmetric proofs of the Atiyah–Singer index theorem. A more formal account of this work, with analytic details, may be found in ref. [16].

The following conventions will be used: even variables, which anticommute with all variables, will be denoted by lower case latin letters, while odd variables, which anticommute with one another but commute with even variables, will be denoted by lower case greek letters. $\mathbb{B}^{m,n}$ will denote the space whose elements are $(m+n)$-tuples $(x^1,\dots,x^m,\theta^1,\dots,\theta^n)$ with $x^1,\dots,x^m$ even and $\theta^1,\dots,\theta^n$ odd. The space of functions of n odd variables $\theta^1,\dots,\theta^n$ of the form

$$f(\theta^1,\dots,\theta^n) = \sum_{\mu\in M_n} f_\mu \theta^{\mu_1}\cdots\theta^{\mu_k}, \tag{1}$$

where each $\mu = \mu_1,\dots,\mu_k$ is a multi-index, with $1 < \mu_1 \le \cdots \le \mu_k < n$, and M_n denotes the set of all such multi-indices (including the empty one), will be denoted $G^\infty(n)$. The coefficients f_μ will take values in some specified space. Integration of functions of anticommuting variables will follow the Berezin prescription [3]

$$\int \mathrm{d}^n\theta\, f(\theta) = f_{1\cdots n}, \tag{2}$$

where $f_{1\cdots n}$ is the coefficient of $\theta^1\cdots\theta^n$ in the expansion (1) of the function f. The integral kernel of an operator H on the function space $G^\infty(n)$ is a function $H(\theta^1,\dots,\theta^n,\phi^1,\dots,\phi^n)$ of $2n$ anticommuting variables such that

$$Hf(\theta) = \int \mathrm{d}^n\phi\, H(\theta,\phi)f(\phi). \tag{3}$$

A useful feature of the Berezin integral is that it allows the trace of an operator to be calculated from its kernel; it can be shown by explicit calculation that

$$\operatorname{tr} H = \int \mathrm{d}^n\theta\, H(\theta,-\theta). \tag{4}$$

Throughout these lectures the language of probability theory will be used for fermionic analogues; however, such analogues do not have all the properties of their classical counterparts. For instance, the Grassmann Wiener measure defined in the next section is not a true measure, but it seems useful to use the same terminology, indicating the departure from convention by the prefix $\mathcal{G}$, so that, for example, the Grassmann Wiener measure is said to be a $\mathcal{G}$-measure.

2. Fermionic Brownian paths

The fermionic analogue of Brownian paths and Wiener measure will now be defined. Letting I denote the closed interval $[0,t]$ of the real line; the Grassmann Wiener measure is a $\mathcal{G}$-measure on $(\mathbb{B}^{0,2n})^I$, the space of paths in $2n$-dimensional anticommuting space. A typical element of this space is denoted $(\theta^1(t),\dots,\theta^n(t),\rho^1(t),\dots,\rho^n(t)|t\in I)$ or $(\boldsymbol{\theta}(t),\boldsymbol{\rho}(t)|t\in I)$. The $\mathcal{G}$-measure is then defined by specifying its finite distributions. That is, suppose that G is a function on $(\mathbb{B}^{0,2n})^I$ which actually only depends on $\boldsymbol{\theta}(t_i),\boldsymbol{\rho}(t_i)$ at a finite set of times $t_i, i=1,\dots,N$ with $0\le t_1<\dots<t_N\le t$, so that

$$G(\boldsymbol{\theta}(t),\boldsymbol{\rho}(t)) = G(\boldsymbol{\theta}(t_1),\boldsymbol{\rho}(t_1),\dots,\boldsymbol{\theta}(t_N),\boldsymbol{\rho}(t_N)). \tag{5}$$

Then

$$\begin{aligned}\mathbb{E}[G] &\stackrel{\text{def}}{=} \int \mathrm{d}\mu_F\, G \\ &\stackrel{\text{def}}{=} \int \mathrm{d}^{nN}\theta\, \mathrm{d}^{nN}\rho\, F_N(\boldsymbol{\theta}^1,\boldsymbol{\rho}^1,\dots,\boldsymbol{\theta}^N,\boldsymbol{\rho}^N) G(\boldsymbol{\theta}^1,\boldsymbol{\rho}^1,\dots,\boldsymbol{\theta}^N,\boldsymbol{\rho}^N),\end{aligned} \tag{6}$$

where

$$\begin{aligned}&F_N(\boldsymbol{\theta}^1,\boldsymbol{\rho}^1,\dots,\boldsymbol{\theta}^N,\boldsymbol{\rho}^N) \\ &\quad= \exp\left[-\left(\boldsymbol{\rho}^1\cdot\boldsymbol{\theta}^1 + \boldsymbol{\rho}^2\cdot(\boldsymbol{\theta}^2-\boldsymbol{\theta}^1) + \cdots + \boldsymbol{\rho}^N\cdot(\boldsymbol{\theta}^N-\boldsymbol{\theta}^{N-1})\right)\right]\end{aligned} \tag{7}$$

with

$$\boldsymbol{\rho}^r\cdot\boldsymbol{\theta}^r = \sum_{i=1}^{n} \rho^{ir}\theta^{ir} \tag{8}$$

for $r=1,\dots,N$. The distribution F_N corresponds to the heuristic fermionic path integral measure $\exp\int_0^t \overline{\psi}(s)\dot{\psi}(s)\,\mathrm{d}s$. The distributions F_N all have weight one, so that they are probability distributions. They also obey the consistency condition

$$\int \mathrm{d}^n\theta^r\,\mathrm{d}^n\rho^r\, F_N(\boldsymbol{\theta}^1,\boldsymbol{\rho}^1,\dots,\boldsymbol{\theta}^N,\boldsymbol{\rho}^N) = F_{N-1}(\boldsymbol{\theta}^1,\boldsymbol{\rho}^1,\dots,\hat{\boldsymbol{\theta}}^r,\hat{\boldsymbol{\rho}}^r,\dots,\boldsymbol{\theta}^N,\boldsymbol{\rho}^N), \tag{9}$$

where the caret indicates omission of an argument. Functions such as G above are called $\mathcal{G}$-random variables. Particular examples are $\theta_s=\theta^i(s)$ and $\rho_s=\rho^i(s)$ for some $i, 0\le i\le n$ and some $s, 0<s<t$. The collection $\{\theta^i_s,\rho^i_s|i=1,\dots,n, s\in I\}$ is called fermionic Brownian motion. More complicated $\mathcal{G}$-random variables, such as $\int_0^t V(\theta_s,\rho_s)\,\mathrm{d}s$, can be defined by a limiting process. One has the Feynman–Kac formula

$$\exp(-Ht)f(\theta) = \mathbb{E}\left[\exp\left(-\int_0^t V_{\mu\nu}\theta_s^\mu(i\rho_s)^\nu\,\mathrm{d}s\right)f(\theta+\theta_t)\right], \tag{10}$$

where $H = \sum_{\mu,\nu \in M_n} V_{\mu\nu}\theta^{\mu}(\partial/\partial\theta)^{\nu}$ [18]. (Note that the free fermionic Hamiltonian is zero.)

This measure can be combined as a direct product with conventional Wiener measure on $\mathbb{R}^m$ to give a measure on the space $(\mathbb{B}^{m,2n})^I$ of paths in the superspace $\mathbb{B}^{m,2n}$.

3. Stochastic calculus

Heuristic derivations of path integral formulae meet greater difficulties when considering (even in a purely bosonic setting) Hamiltonians of the form

$$H = -\tfrac{1}{2}g^{ij}(x)\partial_1\partial_j + h^i(x)\partial_i + V(x). \tag{11}$$

It is possible to handle the direct, time-slicing approach to such Hamiltonians in an analytically rigorous way, but this approach is not always useful, because it requires a knowledge of the very operator one is hoping to study, or of a closely related operator. Much more effective are the techniques of stochastic calculus. These are unfamiliar to many physicists, and so a summary of those aspects of the standard theory which are important in applications to diffusions (or imaginary-time Schrödinger equations) will now be given, before showing how these methods may be extended to fermionic Brownian motion.

Brownian paths $(b_s | s \in I)$ are almost nowhere smooth, but nevertheless sufficiently regular for $\int_0^t f_s \, \mathrm{d}b_s$ to be defined in the following manner: let $f_s = f(\{b_u | u \leq s\})$ with

$$\mathbb{E}\left(\int_0^t |f_s|^2 \, \mathrm{d}s\right) < \infty. \tag{12}$$

(This is a rather loose definition of an adapted stochastic process on Wiener space.) Then

$$\int_0^t f_s \, \mathrm{d}b_s = \lim_{N\to\infty} \sum_{r=0}^{2^N-1} f_{t_r}(b_{t_{r+1}} - b_{t_r}), \tag{13}$$

where $t_r = rt/2^N$. Itô integrals with respect to multi-dimensional Brownian motion may be defined in a similar manner. (A fuller account of this may be found in a number of places; for physicists the work of Simon [20] is an accessible account.)

A crucial formula is the Itô formula for the change of variable. Suppose that $a_t^1, \ldots, a_t^p$ are stochastic integrals on the Wiener space of Brownian paths in $\mathbb{R}^m$ over the time interval I; that is, there exist adapted stochastic processes $f_{a,s}^i, g_s^i, i = 1, \ldots p, a = 1, \ldots m, s \in I$ and random variables $a_0^i, i = 1, \ldots, p$

such that

$$a_s^i = a_0^i + \int_0^s \left(\sum_{a=1}^m f_{a,u}^i \, \mathrm{d}b_u^a + g_s^i \, \mathrm{d}s \right). \tag{14}$$

Then, if F is a sufficiently regular function of p variables, $F(a_t^1, \ldots, a_t^p)$ is also a stochastic integral and

$$\begin{aligned} F(a_s^1, \ldots, a_s^p) - F(a_0^1, \ldots, a_0^p) \\ = \int_0^s \sum_{i=1}^p \sum_{a=1}^m \partial_i F(a_u^1, \ldots, a_u^p)(f_{a,u}^i + g_u^i)\, \mathrm{d}u \\ + \tfrac{1}{2} \sum_{i,j=1}^p \sum_{a=1}^m \partial_i \partial_j F(a_u^1, \ldots, a_u^p) f_{a,u}^i f_{a,u}^j \, \mathrm{d}s. \end{aligned} \tag{15}$$

This formula is proved much as the corresponding formula in conventional calculus is proved; the key estimate is that

$$\mathbb{E}\left[(b_{s+\delta s}^a - b_s^a)(b_{s+\delta s}^c - b_s^c) \right] = \tfrac{1}{2} \delta^{ac} \delta s, \tag{16}$$

which accounts for the presence of the second-order term in (15).

This formula will now be applied to obtain a path integral expression for $\exp(-Ht)$ when H is a second order elliptic operator on $\mathbb{R}^m$ of the form (11). First suppose that functions $e_a^i(x), i = 1, \ldots, m, a = 1, \ldots, p$ satisfy

$$e_a^i(x) e_a^j(x) = g^{ij}(x). \tag{17}$$

(Here and in the remainder of the paper the summation convention that repeated indices are to be summed over their range will be used.) Then consider the stochastic differential equation

$$x_s^i = x^i + \int_0^s \left[e_a^i(x_u)\, \mathrm{d}b_u^a - h^i(x_u)\, \mathrm{d}u \right], \tag{18}$$

where $x^i \in \mathbb{R}^m$. (Such equations are known to have unique solutions x_s, provided that the functions g^{ij} and h^i are sufficiently regular.) Given $f \in \mathrm{C}^\infty$, set

$$F_s(f, x) = \exp\left(- \int_0^s V(x_u)\, \mathrm{d}u \right) f(x_s). \tag{19}$$

Then, applying the Itô formula (15), one obtains

$$\begin{aligned} F_s(f,x) - F_0(f,x) = \int_0^s \Bigg[\exp\left(- \int_0^u V(x_v)\, \mathrm{d}v \right) \Big(-V(x_u) f(x_u)\, \mathrm{d}u \\ + \partial_i f(x_u) \left[e_a^i(x_u) \mathrm{d}b_u^a - h^i(x_u)\, \mathrm{d}u + \tfrac{1}{2} \partial_i \partial_j e_a^i(x_u) e_a^j(x_u)\, \mathrm{d}u \right] \Big) \Bigg]. \end{aligned} \tag{20}$$

If the operator U_s on $C^\infty(\mathbb{R}^m)$ is now defined by setting

$$U_s f(x) = \mathbb{E}(F_s(f,x)), \tag{21}$$

then (using the fact that the expectation of the Itô integral of an adapted function is always zero), one finds that

$$U_s f(x) - f(x) = \int_0^s U_u H f(x)\, \mathrm{d}u, \tag{22}$$

where H is the operator

$$H = -\tfrac{1}{2} g^{ij}(x)\partial_1\partial_j + h^i(x)\partial_i + V(x), \tag{23}$$

and hence

$$U_s = \exp(-Hs), \tag{24}$$

so that the Feynman–Kac formula

$$\exp(-Hs)f(x) = \exp\left(-\int_0^s V(x_u)\, \mathrm{d}u\right) f(x_s) \tag{25}$$

has been established.

4. Fermionic paths and stochastic calculus

Fermionic paths can be incorporated into stochastic calculus, but it is neither necessary nor possible to define integrals along fermionic paths. In the case of standard, bosonic Brownian paths, stochastic integrals allow path integral quantisation techniques to handle Hamiltonians which are arbitrary second-order elliptic operators of the form (11); in the case of fermionic paths, because they are defined in phase space, all derivative operators can be handled by the simple Feynman–Kac formula (10), without the necessity of introducing stochastic integrals. Moreover, it can be seen in a number of ways that fermionic paths are too irregular to allow any simple analogue of the Itô integral; inspection of the Fourier mode analysis of fermionic Brownian paths in ref. [18] indicates that their derivatives would be divergent, while the lack of explicit time dependence in the fermionic Wiener distributions (7) means that increments of all orders remain of order 1, and thus no analogue of the Itô formula (15) exists for fermions unless one includes derivatives of all orders.

However, fermionic paths can be included in the integrand of bosonic Itô integrals, with the corresponding Itô formula being that, if

$$Z_s^i = Z^i + k(\theta_s) + \int_0^s h^i(Z_u, \theta_u, \rho_u)\, \mathrm{d}u + \int_0^t e_a^i(Z_u, \theta_u, \rho_u)\, \mathrm{d}b_u^a, \tag{26}$$

then, for sufficiently regular functions F,

$$F(Z_s) - F(Z_0) =_{\mathbb{E}} \int_0^u \{[h^i(Z_u, \theta_u, \rho_u) + e_a^i(Z_u, \theta_u, \rho_u)\,\mathrm{d}b_u^a]\partial_i F(Z_u)$$

$$+ \tfrac{1}{2} e_j^i(Z_u, \theta_u, \rho_u) e_a^j(Z_u, \theta_u, \rho_u)\partial_i\partial_j F(Z_u)\,\mathrm{d}u\}, \quad (27)$$

where the symbol $=_{\mathbb{E}}$ indicates that two $\mathcal{G}$-random variables have equal expectations. This leads to a Feynman–Kac formula for Hamiltonians which resemble (11) but include fermion operators. Full details may be found in ref. [19].

5. Brownian paths on supermanifolds

In this section the approach to Brownian paths on manifolds which may be found in the work of Elworthy [6], Malliavin [13] and Ikeda and Watanabe [10] will be briefly reviewed and then extended to Brownian paths on supermanifolds.

The theory of geometric Brownian paths on Riemannian manifolds has two components. First, suppose that $V_a, a = 1, \ldots, p$ are vector fields on an m-dimensional manifold M. Then, in local coordinates $x^i, i = 1, \ldots, m$ on some coordinate patch of M, $V_a = V_a^i(x)(\partial/\partial x^i)$. If one considers the stochastic differential equations

$$x_s^i = x^i + \int_0^s [V_a^i(x_u)\,\mathrm{d}b_u^a + \tfrac{1}{2} V_a^j(x_u)\partial_j V_a^i(x_u)\,\mathrm{d}u] \quad (28)$$

(with b^a being p-dimensional Brownian motion), one finds that under change of coordinate $x^i \to \tilde{x}^i(x)$ this stochastic differential equation transforms covariantly—the non-tensorial part of the equation exactly compensates for the second-order term in the Itô formula for the change of variable. The Stratonovich integral allows one to systematise this; given two stochastic integrals X and Y with

$$X_s = \int_0^s [f_{a,u}\,\mathrm{d}b_u^a + f_{0,u}\,\mathrm{d}u],$$

$$Y_s = \int_0^s [g_{a,u}\,\mathrm{d}b_u^a + g_{0,u}\,\mathrm{d}u], \quad (29)$$

the Stratonovich differential is defined by setting

$$Y_s \circ \mathrm{d}X_s \stackrel{\text{def}}{=} Y_s(f_{a,s}\,\mathrm{d}b_s^a + f_{0,s}\,\mathrm{d}s) + \tfrac{1}{2} f_{a,s} g_{a,s}\,\mathrm{d}s. \quad (30)$$

The integrand in (28) can then be expressed as $V_a^i(x_u) \circ \mathrm{d}b_u^a$. Stratonovich integrals have much better transformation properties than Itô integrals, and are thus useful in a geometric context.

The covariance of eq. (28) allows one to solve the equation globally on the manifold (although quite sophisticated patching techniques between different coordinate patches are required). We have seen above that a solution to a stochastic differential equation provides us with a Feynman–Kac formula for a Hamiltonian which may be deduced from the stochastic differential equation. In the case of the stochastic differential equation (28) the corresponding Feynman–Kac formula is

$$\exp(-Hs)f(x) = \mathbb{E}f(x_s) \tag{31}$$

with Hamiltonian

$$\begin{aligned} H &= -\tfrac{1}{2}V_a^i V_a^j \partial_i \partial_j - \tfrac{1}{2}V_a^j \partial_j V_a^i \partial_i \\ &= -\tfrac{1}{2}V_a V_a . \end{aligned} \tag{32}$$

This Hamiltonian is globally defined, as one would expect from the covariance of the corresponding stochastic differential equation.

The second component in the theory of geometric Brownian paths on an m-dimensional Riemannian manifold M is the existence of a canonical set of vector fields $V_a, a = 1, \ldots, m$ on $O(M)$, the bundle of orthonormal frames on M. In local coordinates about the point (x, e_a) of $O(M)$,

$$V_a = e_a^i \partial_i - e_a^i e_b^j \Gamma_{jk}^i \, \partial/\partial e_b^k . \tag{33}$$

These vector fields lead, by the process described above, to a Feynman–Kac formula for the Hamiltonian

$$H = -\tfrac{1}{2}V_a V_a \tag{34}$$

which is the scalar Laplacian when applied to functions on $O(M)$ which depend on x but not on e_a, that is, to functions on M.

By using fermionic paths (in addition to bosonic ones) on a carefully constructed supermanifold, it is possible to extend this approach to obtain a Feynman–Kac formula for the Laplace–Beltrami operator $L = \frac{1}{2}(d + \delta)^2$ on the space of forms on a Riemannian manifold, and to the twisted Laplace–Beltrami operator on forms which take their values in a vector bundle over the manifold.

Given an m-dimensional Riemannian manifold M, together with an n-dimensional Hermitian vector bundle E over M, the required supermanifold $S(E)$ has dimension $(m, m+n)$ and is constructed in the following manner. Suppose that $\{U_\alpha\}$ is an open cover of M by coordinate neighbourhoods which are also trivialisation neighbourhoods of the bundle E, with

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to U(n) \tag{35}$$

the transition functions of the bundle. Then the required supermanifold has local coordinates $x_{(\alpha)}^i, i = 1, \ldots, m, \theta_{(\alpha)}^i, i = 1, \ldots, m, \eta_{(\alpha)}^p, p = 1, \ldots, n$ with coordinate changes on overlapping neighbourhoods being as on the underlying manifold for the even coordinates x^i and, for the odd coordinates,

$$\theta^i_{(\beta)} = \frac{\partial x^i_{(\beta)}}{\partial x^j_{(\alpha)}}\theta^j_{(\alpha)}, \qquad \eta^p_{(\beta)} = h_{\alpha\beta}{}^p{}_q(x_{(\alpha)})\eta^q_{(\alpha)}. \tag{36}$$

(Full details of the patching construction of this supermanifold from its transition functions may be found in ref. [17].)

Now functions on this supermanifold of the form

$$f(x,\theta,\eta) = \sum_{p=1}^{n}\sum_{\mu\in M_n} f_{\mu p}(x)\theta^\mu\eta^p \tag{37}$$

correspond to twisted forms on M with $\theta^\mu \leftrightarrow \mathrm{d}x^\mu$, and the action of a $\mathrm{U}(n)$ matrix $(A^p{}_q)$ represented by $A^p{}_q\eta^q\,\partial/\partial\eta^p$. The Laplace–Beltrami operator $\frac{1}{2}(d+\delta)^2$ can thus be expressed as a differential operator on this space. In fact one can extend the proof of the Weitzenbock formula given by Cycon, Froese, Kirsch and Simon [5] to obtain the twisted Weitzenbock formula

$$\begin{aligned}\tfrac{1}{2}(d+\delta)^2 &= -\tfrac{1}{2}(B - R_i^j(x)\theta^i\delta_{\theta^j} - \tfrac{1}{2}R_{ki}{}^{jl}(x)\theta^i\theta^k\delta_j\delta_l \\ &\quad + \tfrac{1}{4}[\psi^i,\psi^j]F_{ijp}{}^q(x)\eta^p\delta_{\eta^q}),\end{aligned} \tag{38}$$

where B is the twisted Bochner Laplacian

$$B = g^{ij}(D_iD_j - \Gamma^k_{ij}D_k) \tag{39}$$

with

$$D_i = \partial_i + \Gamma^k_{ij}(x)\theta^j\delta_{\theta^k} + A^s_{ir}\eta^r\delta_{\eta^s}, \tag{40}$$

and $\delta_{\theta^i} = \partial/\partial\theta^i, \delta_{\eta^p} = \partial/\partial\eta^p$ and $\psi^i = \theta^i + g^{ij}(x)\delta_{\theta^j}$.

Now let $S(O(M),E)$ denote the supermanifold obtained by including even coordinates in $S(E)$ so that the underlying even manifold is $O(M)$, and consider the vector fields $W_a, a = 1,\ldots,m$ on this supermanifold, where

$$W_a = e^i_a\partial_i - e^j_ae^k_b\Gamma^i_{jk}\,\partial/\partial e^i_b - e^j_a\theta^k\Gamma^i_{jk}\delta_{\theta^i} - e^j_a\eta^rA^s_{jr}\delta_{\eta^s}. \tag{41}$$

Because of the particular nature of the transition functions of $S(O(M),E)$ this defines m vector fields globally on $S(O(M),E)$. Now

$$W_aW_a = B, \tag{42}$$

and thus, if we find stochastic processes $x^i_s, \xi^i_s, e^i_{a,s}, \eta^p_s, i, a = 1,\ldots,m, p = 1,\ldots,n$, which satisfy

$$x^i_s = x + \int_0^s e^i_{a,u}\circ \mathrm{d}b^u,$$

$$e^i_{a,s} = e^i_a + \int_0^s -e^l_{a,u}\Gamma^i_{kl}(x_u)e^k_{b,u}\circ \mathrm{d}b^b_u,$$

$$\xi_s^i = \theta^i + \theta^a e_{a,s}^i + \int_0^s (-\xi_u^j \Gamma_{jk}^i(x_u) e_{b,u}^k \circ \mathrm{d}b_u^b$$
$$-\theta_u^a \, \mathrm{d}e_{a,u}^i + \tfrac{1}{4}\mathrm{i}\xi_u^j R^i{}_{jk\ell}(x_u)\xi_u^k \pi_u^\ell \, \mathrm{d}u),$$
$$\eta_s^p = \eta^p + \int_0^s \left[-e_{a,u}^j \eta_u^q A_{jq}^p(x_u) \circ \mathrm{d}b_u^a\right.$$
$$\left. + \tfrac{1}{4}\eta_u^q(\xi_u^i + \mathrm{i}\pi_u^i)(\xi_u^j + \mathrm{i}\pi_u^j) F_{ijq}{}^p(x_u)\, \mathrm{d}u\right], \tag{43}$$

where

$$\pi_s^i = e_{a,s}^i \rho_s^a, \tag{44}$$

then one obtains the Feynman–Kac formula

$$\exp(-Ls) f(x,\theta,\eta) = \mathbb{E}\left[f(x_s, \xi_s, \eta_s)\right], \tag{45}$$

where L is the twisted Laplace–Beltrami operator $\frac{1}{2}(d+\delta)^2$ and f is a function of the form (37).

6. Applications to geometry

As an illustration of the analytic power of this techniques, the Atiyah–Singer index theorem for the twisted Hirzebruch signature theorem will now be proved. (It was shown by Atiyah, Bott and Patodi [2] that the full theorem follows from this special case by K-theoretic arguments.) The proof is a rigorous version of the supersymmetric proofs of the index theorem introduced by Alvarez-Gaumé [1] and by Friedan and Windey [8]. Various other authors have used probabilistic methods to prove the index theorem [4,7,11,12,21], but these works do not use the fermionic paths described in this paper, which are a rigorous version of the paths used in refs. [1] and [8].

The proof makes use of the McKean and Singer formula for the index I of the twisted Hirzebruch signature complex [15],

$$I = \operatorname{Tr} \gamma^5 \exp(-Lt), \tag{46}$$

where L is the Laplace–Beltrami operator $\frac{1}{2}(d+\delta)^2$ as before, and Tr denotes a full trace over both operators and matrices. The result to be proved is that

$$I = \int_M \left[\operatorname{tr} \exp\left(\frac{-F}{2\pi}\right) \det\left(\frac{\mathrm{i}\Omega/2\pi}{\tanh \mathrm{i}\Omega/2\pi}\right)^{1/2}\right], \tag{47}$$

where F is the curvature two-form of the connection A on the bundle E, Ω is the curvature connection of the Riemannian connection on (M, g), and the square brackets indicate projection onto the m-form component. In fact a stronger, local result will be proved, that is, it will be shown that at each point p of the manifold M

$$\lim_{t\to 0} \operatorname{tr} \gamma^5 \exp(-Lt)(p,p) = \left[\operatorname{tr} \exp\left(\frac{-F}{2\pi}\right) \det\left(\frac{\mathrm{i}\Omega/2\pi}{\tanh \mathrm{i}\Omega/2\pi}\right)^{1/2} \right]_p , \quad (48)$$

where tr denotes a matrix trace.

To evaluate the left hand side of (48), the Feynman–Kac formula (45) must be used. Additionally, several steps in the analysis of Grassmann variables are required, and Du Hamel's formula must be used to extract information about the kernel of $\exp(-Lt)$ from information about the action of the operator on functions. The most important difference between the approach used in heuristic physicists' calculations of path integrals in curved space and the approach used here is that stochastic differential equations are used, which means that different paths are used but the measure on path space is unchanged, whereas in the heuristic approach it is the measure which is changed, which is a much harder task to handle analytically.

Beginning with the anticommuting variables, the first step is to observe that the action of γ^5 on twisted forms is

$$\gamma^5 \sum_{\mu\in M_n} \sum_{p=1}^{n} f_{\mu p}(x)\theta^\mu \eta^p$$
$$= \sum_{\mu\in M_n} \sum_{p=1}^{n} \int \mathrm{d}^m \rho \frac{1}{\sqrt{\det g^{ij}}} \exp(\mathrm{i}\rho^i \theta^j) g_{ij}(x) f_{\mu p}(x) \rho^\mu \eta^p. \quad (49)$$

The operator γ^5 is an involution, which resembles the Hodge star operator.

As we are working at one particular point $p \in M$ we will use normal coordinates x^i about this point, so that p has coordinates 0, while

$$\begin{aligned} g^{ij}(x) &= \delta^{ij} - \tfrac{1}{3} x^k x^\ell R_k{}^i{}_\ell{}^j(0) + \text{h.o.t.}, \\ \det(g_{ij}(x)) &= 1 + \text{h.o.t.}, \\ \Gamma^k_{ij} &= \tfrac{1}{3} x^\ell (R_{\ell ji}{}^k(0) + R_{\ell ij}{}^k(0)) + \text{h.o.t.}, \\ A^s_{ir} &= -\tfrac{1}{2} x^j F_{ijr}{}^s(0) + \text{h.o.t.}, \end{aligned} \quad (50)$$

where h.o.t. denotes higher-order terms. It is shown by Cycon et al. [5] that for the purposes of calculating $\lim_{t\to 0} \operatorname{tr} \gamma^5 \exp(-Lt)(p,p)$ one may replace the manifold M with $\mathbb{R}^m$ and the bundle E over M with the trivial bundle $\mathbb{R}^m \times \mathbb{C}^n$ provided that the metric and connection are chosen to agree with those on the manifold on a neighbourhood of p (identified with a neighbourhood of 0 in $\mathbb{R}^m$ by the coordinate functions) and to be the Euclidean metric and the zero connection outside some compact region of $\mathbb{R}^m$. Thus from now on these replacements will be made.

Using the expression (49) for the action of γ^5, together with the expression (4) for the trace of an operator in terms of a Berezin integral, one finds that

$$\operatorname{tr}\gamma^5 \exp(-Lt)(0,0) = \int \mathrm{d}^m\rho\, \mathrm{d}^m\theta\, \mathrm{d}^n\eta\, \exp(-Lt)(0,0,\rho,-\theta,\eta,-\eta)\exp(\mathrm{i}\rho\cdot\theta). \quad (51)$$

Now the Feynman–Kac formula (45) tells us about the action of $\exp(-Lt)$ on functions. Using Du Hamel's formula [9] one may extract information about the kernel of this operator. Suppose that H_0 is the operator $-\frac{1}{2}\partial_i\partial_i$ on $C^\infty(\mathbb{R}^m)$. Then Du Hamel's formula states that

$$\exp(-Lt)(x,x',\theta,\theta',\eta,\eta') - \exp(-H_0t)(x,x',\theta,\theta',\eta,\eta') = \int_0^t \exp[-L(t-s)](L-H_0)\exp[-H_0s(x,x',\theta,\theta',\eta,\eta')], \quad (52)$$

where the operators L and H_0 act with respect to the variable x,θ and η. Taking the supertrace of both sides (that is, operating with γ^5 and taking the trace) one finds that

$$\operatorname{tr}\gamma^5\exp(-Lt)(0,0) = \int \mathrm{d}^m\theta\,\mathrm{d}^m\rho\,\mathrm{d}^n\eta \int_0^t \Big\{\exp[-L(t-s)](L-H_0) \times \Big(\exp[-H_0s(x,0,\rho,\theta,\eta,\pi)]\Big)\Big|_{x=0,\pi=-\eta}\exp(-\theta\cdot\rho)\Big\} \quad (53)$$

[using the fact that $\operatorname{tr}\gamma^5\exp(-H_0t) = 0$].

Now

$$\exp(-H_0t)(x,0,\theta,\theta',\eta,\eta') = \int \mathrm{d}^m\rho\,\mathrm{d}^n\kappa\,(2\pi t)^{-m/2}\exp(x^2/2t) \times \exp[-\mathrm{i}\rho\cdot(\theta-\theta')]\exp[-\mathrm{i}\kappa(\eta-\eta')], \quad (54)$$

so that (using the Feynman–Kac formula (45), and carrying out some integration)

$$\operatorname{tr}\gamma^5\exp(-Lt)(0,0) = \mathbb{E}\Big[\int \mathrm{d}^m\theta\,\mathrm{d}^n\eta\,\mathrm{d}^n\kappa \times \int_0^t \mathrm{d}s\,(2\pi s)^{-m/2}F_s(x_{t-s},\xi_{t-s},\theta,\eta_{t-s},\kappa)\exp(-\kappa\cdot\eta)\Big], \quad (55)$$

where

$$F_s(x,\theta,\rho,\eta,\kappa) = (L-H_0)\exp(-x^2/2s)\exp(-\mathrm{i}\rho\cdot\theta)\exp(-\mathrm{i}\kappa\cdot\eta). \quad (56)$$

Now this can be estimated as t tends to 0 in the following way. First let x^{1i}_s, ξ^{1i}_s, η^{1p}_s satisfy the stochastic differential equations

$$x^{1i}_s = b^i_s,$$

$$\xi^1{}^i_s = \theta^i + \theta^a_s\delta^i_a + \int_0^s \left[\tfrac{1}{3}\xi^1{}^j_u x^1{}^l_u (R_{lkj}{}^i + R_{ljk}{}^i)\,\mathrm{d}b^k_u \right.$$
$$\left. + \tfrac{1}{3}\xi^1{}^j_u R^i_j\,\mathrm{d}u - \tfrac{1}{4}\mathrm{i}\xi^1{}^j_u\xi^1{}^k_u\rho^l_u R_{jk}{}^i{}_l\,\mathrm{d}u\right],$$
$$\eta^1{}^p_s = \eta^p + \int_0^s \tfrac{1}{4}\eta^1{}^q_u(\theta^i_u - \mathrm{i}\pi^i_u)(\theta^j_u - \mathrm{i}\pi^j_u)F_{ijq}{}^p\,\mathrm{d}u. \tag{57}$$

(Here $R_{ijk}{}^l = R_{ijk}{}^l(0)$, and indices are raised and lowered by $g^{ij}(0) = \delta^{ij}$.) Now solutions to this set of stochastic differential equations are close to those of (43). In fact $x_s - x^1{}_s \sim \sqrt{s^3}, \xi_s - \xi^1{}_s \sim s, \eta_s - \eta^1{}_s \sim s$ and $e^i_{a,s} - \delta^i_a \sim s$ [16]. Thus

$$\lim_{t\to 0}\operatorname{tr}\gamma^5\left[\exp(-Lt)(0,0) - \exp(-L^1t)(0,0)\right] = 0, \tag{58}$$

where L^1 is the Hamiltonian corresponding to the simplified stochastic differential equations (57).

This simpler Hamiltonian L^1 can be handled by flat space path integral methods. Use of these, together with further use of Du Hamel's formula, shows that

$$\begin{aligned}&\lim_{t\to 0}\operatorname{tr}\gamma^5\exp(-L^1t)(0,0)\\ &= \lim_{t\to 0}\int \mathrm{d}^m\rho\,\mathrm{d}^m\theta\,\mathrm{d}^n\eta\,\exp(-L^1t)(0,0,\rho,-\theta,\eta,-\eta)\exp(\mathrm{i}\rho\cdot\theta)\\ &= \lim_{t\to 0}\int \mathrm{d}^m\rho\,\mathrm{d}^m\theta\,\mathrm{d}^n\eta\,\exp(-L^2t)(0,0,\rho,-\theta,\eta,-\eta)\exp(\mathrm{i}\rho\cdot\theta),\end{aligned} \tag{59}$$

where

$$L^2 = H_x + H_\theta + H_\eta, \tag{60}$$

with

$$\begin{aligned}H_x &= -\left(\tfrac{1}{2}\partial_i\partial_i + \tfrac{1}{2}\hat{x}^l\frac{\phi^j}{\sqrt{2\pi t}}\frac{\phi^i}{\sqrt{2\pi t}}R_{jil}{}^k\partial_k \right.\\ &\qquad \left. + \tfrac{1}{8}\hat{x}^k\hat{x}^l\frac{\phi^i\phi^j\phi^{n'}\phi^{m'}}{(2\pi t)^2}R_{n'ikp}R_{m'jl}{}^p\right),\\ H_\theta &= -\tfrac{1}{4}R_{ijkl}\frac{\phi^i\phi^j}{2\pi t}\psi^k\psi^l,\\ H_\eta &= -\frac{\phi^i}{\sqrt{2\pi t}}\frac{\phi^j}{\sqrt{2\pi t}}F_{ijp}{}^q\hat{\eta}^p\delta_{\eta^q},\end{aligned} \tag{61}$$

with $\phi = \sqrt{t}\theta$. (This rescaling allows one to pick out those terms which survive in the limit as t tends to zero. Note that $\mathrm{d}\theta = \sqrt{t}\,\mathrm{d}\phi$.) Now $\exp[-H_x t(0,0)]$ can be evaluated using the result given by Simon [20] for $\mathbb{R}^2$ that, if

$$H = -\tfrac{1}{2}\partial_i\partial_i + \tfrac{1}{2}\mathrm{i}B(x^1\partial_2 - x^2\partial_1) + \tfrac{1}{8}B^2((x^1)^2 + (x^2)^2), \tag{62}$$

then

$$\exp[-Ht(0,0)] = \frac{B}{4\pi\sinh(\frac{1}{2}Bt)}. \tag{63}$$

Thus, if $\Omega_k{}^l = \frac{1}{2}\phi^i\phi^j R_{ijk}{}^l$ is regarded as an $m \times m$ matrix, skew-diagonalised as

$$(\Omega_k{}^l) = \begin{pmatrix} 0 & \Omega_1 & \dots & 0 & 0 \\ -\Omega_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \Omega_{m/2} \\ 0 & 0 & \dots & -\Omega_{m/2} & 0 \end{pmatrix}, \tag{64}$$

then

$$\exp[-H_x t(0,0)] = \prod_{k=1}^{m/2} \frac{\mathrm{i}\Omega_k}{2\pi t}\frac{1}{\sinh(\mathrm{i}\Omega_k/2\pi)}. \tag{65}$$

Also, using fermion paths [18] or direct calculation,

$$\exp[-tH_\theta(\theta,\theta')] = \int \mathrm{d}^m\rho\Big(\exp[-\mathrm{i}\rho(\theta-\theta')]$$

$$\prod_{k=1}^{m/2}\left[\cosh(\mathrm{i}\Omega_k/2\pi) + (\theta^{2k-1}+\mathrm{i}\rho^{2k-1})(\theta^{2k}+\mathrm{i}\rho^{2k})\sinh(\mathrm{i}\Omega_k/2\pi)\right]\Big). \tag{66}$$

Thus

$$\operatorname{tr}\gamma^5\exp[-Ht(0,0)]$$

$$= \int \mathrm{d}^m\phi \prod_{k=1}^{m/2}\frac{\mathrm{i}\Omega_k}{2\pi}\frac{1}{\sinh(\mathrm{i}\Omega_k/2\pi)}\cosh\left(\frac{\mathrm{i}\Omega_k}{4\pi}\right)\operatorname{tr}\exp\left(-\phi^i\phi^j\frac{F_{ij}}{2\pi}\right). \tag{67}$$

Hence

$$\operatorname{tr}\gamma^5\exp[-Lt(p,p)] = \left[\operatorname{tr}\exp\left(\frac{-F}{2\pi}\right)\det\left(\frac{\mathrm{i}\Omega/2\pi}{\tanh \mathrm{i}\Omega/2\pi}\right)^{1/2}\right]\Bigg|_p, \tag{68}$$

as required.

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